As stated earlier, $p_b = p_\infty$ is assumed.

From Fig. 2 we observe that C_{D_j} rises sharply with an increase in jet momentum, and at $C_F = 0.23$ becomes equal to the drag coefficient resulting from pressure distribution (C_{D_p}). Beyond this point C_{D_j} is higher than C_{D_p}. Unfortunately, enough experimental data are not available for larger values of C_F. However, from the only available data at $C_F = 0.25$ (Fig. 3), we observe that the forebody drag coefficient at this value of C_F is higher than that at $C_F = 0.23$. Thus for any further increase in C_F, the trend will reverse and the forebody drag coefficient starts increasing. Therefore it can be inferred that the maximum achievable forebody drag reduction using this technique is limited to 50% by employing C_F around 0.23-0.25.

From Fig. 3 we note that the forebody drag coefficients deduced from Warren[1] and Finley's[3] experimental data correlate well in terms of a single parameter C_F. The deviation observed at higher values of C_F is apparently caused by the reversal of trend mentioned earlier. The correlation can be expressed mathematically as

$$C_D = \tfrac{1}{2} C_{D_0} C_F^{-1/6}, \quad 0.015 < C_F < 0.25 \tag{3}$$

The forebody drag reduction due to upstream facing jet is very similar to that of a protruding solid spike from the leading edge of a blunt body. The problem of a blunt body with leading edge spike at high speeds has been studied by a number of investigators, and their work has been discussed in detail by Chang.[5] A thin protruding probe, called a spike, mounted in front of a blunt body moving at high speeds is capable of reducing the forebody drag due to flow separation on the probe. The flow separation occurs because the boundary layer on the probe interacts with the shock wave generated at the junction with the body. In front of the body and between the oblique shock wave and the probe, a conical region of separated flow is formed in which the pressure is greatly reduced, as in the present problem. From the results presented by Chang,[5] it is seen that a forebody drag reduction of up to 50% can be achieved. The forebody drag coefficient of a spiked body is not very sensitive to Mach or Reynolds number.

In summary, we observe that the mechanisms of forebody drag reduction either by a spike or by a forward facing jet are quite similar. The maximum drag reductions achievable are also of the same order. However, a forward facing jet has the additional capability of reducing aerodynamic heating, which can be quite severe at high Mach numbers. Using the correlation presented in this work, the ejection parameters can be chosen to achieve maximum permissible forebody drag reduction by the technique suggested here.

References

[1] Warren, C.H.E., "An Experimental Investigation of the Effect of Ejecting a Coolant Gas at the Nose of a Bluff Body," *Journal of Fluid Mechanics*, Vol. 8, 1960, pp. 400-417.

[2] Romeo, D. J. and Sterret, D. R., "Exploratory Investigation of the Effect of a Forward-Facing Jet on the Bow Shock of a Blunt Body in a Mach Number 6 Free Stream," NASA TN D-1605, Feb. 1963.

[3] Finley, P. J., "The Flow of a Jet From a Body Opposing a Supersonic Free Stream," *Journal of Fluid Mechanics*, Vol. 26, Pt. 2, 1966, pp. 337-368.

[4] Hoerner, S. F., *Fluid Dynamic Drag*, Published by the author, 1965, pp. 18-13 to 18-24.

[5] Chang, P. K., "Separation of Flow," *International Series of Monographs in Interdisciplinary and Advanced Topics in Science and Engineering*, Vol. 3, Pergamon Press, New York, 1970, pp. 452-530.

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Potential of Transformation Methods in Optimal Design

Ashok D. Belegundu* and Jasbir S. Arora†

The University of Iowa, Iowa City, Iowa

I. Introduction

THE purpose of this Note is to present a general framework for the efficient use of transformation methods in optimal design of large structural and mechanical systems. The term "transformation method" is used to describe any method that solves the constrained optimization problem by transforming it into one or more unconstrained optimization problems.

A basic difficulty in optimal structural design is that the evaluation of constraint functions and their gradients is very expensive. This is because many constraints of the design problem are implicit functions of design variables. The transformation methods essentially collapse all constraints of the design problem into one equivalent constraint[1] which serves as a penalty term for the transformation methods. This is very desirable for structural design applications, since it implies that one has to calculate the derivative of only one implicit function as compared to the calculation of derivatives of several implicit functions in other methods. Therefore, transformation methods can result in substantial computational savings in optimal design of structural and mechanical systems.

There are three classes of transformation methods that have been discussed in the literature: the penalty function, the barrier function, and the multiplier (or augmented Lagrangian) methods. The penalty and barrier methods have been referred to as sequential unconstrained minimization techniques (SUMTs) by Fiacco and McCormick.[2] SUMTs possess a number of undesirable properties. The weaknesses are most serious when a controlling parameter r is large. For large r, the penalty and barrier functions are ill-behaved near the boundary of the constraint set where the optimum points usually lie. Furthermore, it is shown[3,4] that the Hessian matrix of the unconstrained function becomes ill-conditioned as $r \to \infty$. In spite of these difficulties, SUMTs have been successfully used for many structural and mechanical system design applications. Notable among this body of literature are the works of Schmit[5] and Haftka.[6] More references on works of these authors may be found in Refs. 5 and 6.

The multiplier methods have been developed in the recent literature to alleviate some of the numerical difficulties encountered in SUMTs. These methods do not require large values of the controlling parameters, and also possess better convergence properties than SUMTs.[7]

II. Optimal Design Problem and Computation of Derivatives

To present ideas of the transformation methods as they apply to the structural and mechanical system design, a simplified model of the design problem is considered. It is understood that the methods presented here are applicable to more complex models for the design problem.[5,6,8]

The optimal design problem is defined as follows: minimize

$$\psi_0(b,z) \tag{1}$$

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*Research Assistant, Division of Materials Engineering. College of Engineering.

†Professor, Division of Materials Engineering. College of Engineering.

subject to

$$K(b)z=S(b) \tag{2}$$

$$\psi_i(b,z)\leq 0 \qquad i=1,2,\ldots,m \tag{3}$$

where b is a k vector of design variables, z an n vector of nodal displacements, n the number of degrees of freedom of the finite-element model for the structure, $K(b)$ an $n\times n$ structural stiffness matrix (symmetric and positive definite), and $S(b)$ an n vector of applied loads. The functions $\psi_i(b,z)$ represent constraints (stress, displacement, and others) and $\psi_0(b,z)$ represents a cost function for the design problem.

There are basically two methods[8,9] for computing derivatives of the implicit functions $\psi_i(b,z(b))$. To keep the discussion general, consider the problem of determining derivatives of a function $q(b,z(b))$. The total derivative of q is given as

$$\underset{(1\times k)}{\frac{dq}{db}} = \underset{(1\times k)}{\frac{\partial q}{\partial b}} + \underset{(1\times n)}{\frac{\partial q}{\partial z}} \; \underset{(n\times k)}{\frac{dz}{db}} \tag{4}$$

where the notation

$$\frac{dq}{db}=\left[\frac{dq}{db_1},\ldots,\frac{dq}{db_k}\right] \quad \text{and} \quad \frac{\partial q}{\partial b}=\left[\frac{\partial q}{\partial b_1},\ldots,\frac{\partial q}{\partial b_k}\right]$$

has been used. In the above equation, arguments of q are suppressed for convenience. By differentiating Eq. (2), one obtains

$$K\frac{dz}{db}+\frac{\partial}{\partial b}(Kz)=\frac{dS}{db}$$

Therefore,

$$\frac{dz}{db}=-K^{-1}\left[\frac{\partial}{\partial b}(Kz)-\frac{dS}{db}\right]\equiv -K^{-1}A \tag{5}$$

By substituting Eq. (5) into Eq. (4), one obtains

$$\frac{dq}{db}=\frac{\partial q}{\partial b}-\frac{\partial q}{\partial z}K^{-1}A \tag{6}$$

In Eq. (6), the computation of K^{-1} is involved, which is a very inefficient operation. There are two ways of avoiding this computation resulting in two methods for computing derivatives of $q(b,z(b))$.[9] The method that is most suitable for the present application is summarized as follows.

Define an n vector λ^q (called the adjoint vector) to be the solution of the linear system

$$K\lambda^q=\partial q^T/\partial z \tag{7}$$

Substituting Eq. (7) into Eq. (6) and using symmetry of K, the derivative of $q(b,z(b))$ is given as

$$\frac{dq}{db}=\frac{\partial q}{\partial b}-\lambda^{q^T}A \tag{8}$$

To keep presentation of the transformation methods simple as they apply to optimal design problems, and for notational convenience, the optimal design problem is restated as: Minimize

$$f(b) \tag{9}$$

subject to

$$g_i(b)\leq 0, \quad i=1,2,\ldots,m \tag{10}$$

It is understood that $f(b)$ and $g_i(b)$ are implicit functions of design variables and their gradients, whenever needed, may be calculated using the method described above.

III. Transformation Methods

In this section basic transformation methods for the optimal design problem of Eqs. (9) and (10) are briefly described. For more details on these methods, see Refs. 2-6, 7, and 10.

All transformation methods convert the constrained problem of Eqs. (9) and (10) into an unconstrained problem for the function

$$\Phi(b,r)=f(b)+P(g(b),r) \tag{11}$$

where r, in general, is a vector of controlling parameters and P is a real-valued function whose action of imposing the penalty is controlled by r. The local minimum of $\Phi(b,r)$ at the νth iteration is denoted by $b^{(\nu)}$. The idea of transformation methods is very attractive because efficient unconstrained algorithms (e.g., Davidon-Fletcher-Powell method) can be used to completely solve the constrained problem. It will be shown in Sec. IV that the idea of transformation methods is also very attractive for design problems where implicit constraints must be treated.

In the penalty function methods, the most popular penalty function $P(g(b),r)$ is given as

$$P(g(b),r)=r\sum_{i=1}^{m}[g_i^+(b)]^2 \tag{12}$$

where r is a scalar parameter, and $g_i^+(b)$ is equal to $\max(0,g_i(b))$. In case of barrier function methods, a popular barrier function $P(g(b),r)$ is given as

$$P(g(b),r)=\frac{1}{r}\sum_{i=1}^{m}\frac{-1}{g_i(b)} \tag{13}$$

where r is again a scalar parameter. Other penalty and barrier functions have been used quite often.[2-6] For both of these methods, it can be shown that as $r^{(\nu)}\to\infty$, $b^{(\nu)}\to b^*$, where b^* is a relative minimum of the constrained problem.

In multiplier methods the function $P(g(b),r)$ is given as[4]

$$P(g(b),r)=\frac{1}{2}\sum_{i=1}^{m}r_i[(g_i(b)+\theta_i)^+]^2 \tag{14}$$

where θ_i is a parameter associated with the ith constraint. During the unconstrained minimization, parameters θ_i are kept constant. After each unconstrained minimization they are changed according to the expression[4]:

$$\theta_i^{(\nu+1)}=\theta_i^{(\nu)}+\max[g_i(b^{(\nu)}),-\theta_i^{(\nu)}] \quad i=1,2,\ldots,m \tag{15}$$

If $\theta_i=0$ and $r_i=r$, then the regular penalty function method is obtained, where convergence is obtained by letting $r\to\infty$. However, the objective of multiplier methods is to keep each $r_i^{(\nu)}=r^{(\nu)}$ (a scalar) fixed at a finite value and change θ_i after each unconstrained minimization.

A general algorithm based on the multiplier methods is given as follows:

Step 1. Set $\nu=0$. Estimate $\theta_i^{(0)}$ and a scalar parameter $r^{(0)}$.

Step 2. Minimize $\Phi(b,\theta^{(\nu)},r^{(\nu)})$ for the multiplier method. Let $b^{(\nu)}$ be the minimum point.

Step 3. Check if $(b^{(\nu)},\theta^{(\nu)},r^{(\nu)})$ satisfy the convergence criteria. If they do, then stop; otherwise, go to step 4.

Step 4. Set $\nu=\nu+1$. Change $\theta_i^{(\nu)}$ according to Eq. (15). Increase $r^{(\nu)}$ appropriately[4] and go to step 2.

It should be noted that any method for unconstrained minimization may be used in step 2. Most commonly used methods require gradients of cost and constraint functions. One-dimensional minimizations (accurate or inaccurate) are used to determine the step size.

IV. Optimal Structural Design with Transformation Methods

The major discussion of the transformation methods as they apply to the structural design problem centers around step 2 of the algorithm. Specifically, the major point of departure of the proposed methods is the way in which the gradient of the transformed function is calculated, and the way in which one-dimensional search is performed in step 2. In the nonlinear programming literature, it is suggested to calculate the gradient of the transformed function from gradients of the cost and constraint functions. For example, for the penalty function of Eq. (12), the gradient of the transformed function Φ is calculated as

$$\frac{d\Phi}{db} = \frac{df}{db} + 2r\sum_{i=1}^{m} g_i^+ \frac{dg_i^+}{db} \tag{16}$$

For the nonlinear programming problem, Eq. (16) presents no particular difficulty as $g_i(b)$ are explicit functions of b and gradient calculations for each constraint are quite simple. However, for the structural design problem, $g_i(b)$ are implicit functions of b. Therefore, the gradient for each constraint must be calculated according to Eq. (8). These gradient calculations are quite expensive and must be avoided.

It is possible to calculate the gradient of the transformed function $\Phi(b,r)$ directly by using the adjoint variable approach described in the previous section. Following that approach,

$$\frac{d\Phi}{db} = \frac{\partial\Phi}{\partial b} - \lambda^{\Phi^T} A \tag{17}$$

where λ^Φ for the function of Eq. (11) is given as the solution of the equation

$$K\lambda^\Phi = \partial\Phi^T/\partial z \tag{18}$$

Calculations for $\partial\Phi/\partial z$ are quite straightforward since Φ is an explicit function of the z variables. For example, using the penalty function of Eq. (12), $\partial\Phi/\partial z$ is given as

$$\frac{\partial\Phi}{\partial z} = \frac{\partial f}{\partial z} + 2r\sum_{i=1}^{m} g_i^+ \frac{\partial g_i^+}{\partial z} \tag{19}$$

The advantage of using Eq. (17) over Eq. (16) to calculate $d\Phi/db$ is obvious. To use Eq. (16), one will have to calculate several adjoint variables to calculate each dg_i^+/db, whereas to use Eq. (17), one needs to calculate only one adjoint variable. It should be noted, however, that both Eqs. (16) and (17) give identical values for the derivatives $d\Phi/db$. To see this, one can substitute values for df/db and dg_i/db from equations similar to Eq. (8), into Eq. (16). Use of Eqs. (18) and (19) then gives the desired equivalence for Eqs. (16) and (17).

A disadvantage of the one adjoint variable approach in calculation for $d\Phi/db$ is that gradients of individual active constraints are not available. Gradients of individual constraints have been used to perform approximate reanalysis of the system in estimating the function Φ during the one-dimensional search for step length determination.[5,6] However, when adjoint variable approach is used to directly calculate the gradient of Φ, exact value of Φ during step size determination can be calculated using Eq. (11) directly. This does involve evaluation of constraints for each increment of the step size parameter implying that analysis of the system must be performed for each change in design. This may be expensive and one may have to resolve to inaccurate one-dimensional searches.

V. Discussion and Conclusions

A basic framework in which transformation methods may efficiently be applied to the optimal design problem is presented. Among the transformation methods, penalty and barrier function methods have been extensively used for optimal design of several classes of structural systems. The multiplier method has not been fully explored for structural design problems, although Imai[11] has used the method for configurational optimization of truss structures. The new idea presented herein relative to the use of transformation methods in structural design is that the adjoint variable approach must be used to calculate the gradient of the transformed function. This can offer substantial computational advantage for design problems where many constraints are implicit functions of the variables.

The basic idea of the multiplier method is viewed as a technique that essentially collapses all constraints of the problem to an equivalent functional constraint.[1] This is very attractive from a computational standpoint and immediately offers several areas for potential applications. One such area is the design of structural systems for damage tolerance.[12] For such design problems, the number of constraints is quite large and the systematic reduction of constraints offered by the multiplier method can be highly useful. Similar remarks can be offered for the dynamic response problems.[8]

It is also noted here that the adjoint vector approach can be extended to calculate second-order derivatives of an implicit function.[13] Therefore, the analysis presented here offers the possibility of using a second-order method (Newton's method) for optimal design of complex engineering systems. The approach that may be followed here is to use regular algorithms without second-order information for the first few iterations. One can then switch to Newton's algorithm for faster terminal convergence.

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References

[1] Arora, J. S., Haug, E. J., and Rajan, S. D., "Efficient Constraint Treatment in Structural Optimization," Paper 80-501, ASCE National Convention, Hollywood, Fla., Oct. 1980.

[2] Fiacco, A. V. and McCormick, G. P., *Nonlinear Programming: Sequential Unconstrained Minimization Techniques,* John Wiley and Sons, Inc., New York, 1968.

[3] Luenberger, D. G., *Introduction to Linear and Nonlinear Programming.* Addison-Wesley, Reading, Mass., 1973.

[4] Gill, P. E. and Murray, W., Eds., *Numerical Methods for Constrained Optimization,* Academic Press, New York, 1974.

[5] Cassis, J. H. and Schmit, L. A., "On Implementation of the Extended Interior Penalty Function," *International Journal for Numerical Methods in Engineering,* Vol. 10, 1976, pp. 3-23.

[6] Prasad, B. and Haftka, R. T., "A Cubic Extended Interior Penalty Function for Structural Optimization," *International Journal for Numerical Methods in Engineering,* Vol. 14, No. 9, 1979, pp. 1107-1126.

[7] Bertsekas, D. P., "Multiplier Methods: A Survey," *Automatica,* Vol. 12, 1976, pp. 133-145

[8] Haug, E. J. and Arora, J. S., *Applied Optimal Design: Mechanical and Structural Systems,* Wiley-Interscience, New York, 1979.

[9] Arora, J. S. and Haug, E. I., "Methods of Design Sensitivity Analysis in Structural Optimization," *AIAA Journal,* Vol. 17, Sept. 1979, pp. 970-974.

[10] Rockafeller, R. T., "Augmented Lagrange Multiplier Functions and Duality in Non-Convex Programming," *SIAM Journal of Control,* Vol. 12, 1974, pp. 268-285.

[11] Imai, K., "Configuration Optimization of Trusses by the Multiplier Method," Ph.D. Dissertation, Rept. UCLA-ENG-7842, 1978.

[12] Arora, J. S., Haskell, D. F., and Govil, A. K., "Optimal Design of Large Structures for Damage Tolerance," *AIAA Journal,* Vol. 18, May 1980, pp. 563-570.

[13] Haug, E. J., "Second Order Design Sensitivity Analysis of Structural Systems," *AIAA Journal,* Submitted Jan. 1981.